

A cohomological description of connections and curvature over posets

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Abstract

What remains of a geometrical notion like that of a principal bundle when the base space is not a manifold but a coarse graining of it, like the poset formed by a base for the topology ordered under inclusion? Motivated by finding a geometrical framework for developing gauge theories in algebraic quantum field theory, we give, in the present paper, a first answer to this question. The notions of transition function, connection form and curvature form find a nice description in terms of cohomology, in general non-Abelian, of a poset with values in a group G . Interpreting a 1-cocycle as a principal bundle, a connection turns out to be a 1-cochain associated in a suitable way with this 1-cocycle; the curvature of a connection turns out to be its 2-coboundary. We show the existence of nonflat connections, and relate flat connections to homomorphisms of the fundamental group of the poset into G . We discuss holonomy and prove an analogue of the Ambrose-Singer theorem.

1 Introduction

One of the outstanding problems of quantum field theory is to characterize gauge theories in terms of their structural properties. Naturally, as gauge theories have been successful in describing elementary particle physics, there is a notion of a gauge theory in the framework of renormalized perturbation

theory. Again, looking at theories on the lattice, there is a well defined notion of a lattice gauge theory.

This paper is a first step towards a formalism which adapts the basic notions of gauge theories to the exigencies of algebraic quantum field theory. If successful, this should allow one to uncover structural features of gauge theories. Some earlier ideas in this direction may be found in [10].

In mathematics, a gauge theory may be understood as a principal bundle over a manifold together with its associated vector bundles. For applications to physics, the manifold in question is spacetime but, in quantum field theory, spacetime does not enter directly as a differential manifold or even as a topological space. Instead, a suitable base for the topology of spacetime is considered as a partially ordered set (*poset*), ordered under inclusion. This feature has to be taken into account to have a variant of gauge theories within algebraic quantum field theory. To do this we adopt a cohomological approach. After all, a principal fibre bundle can be described in terms of its transition functions and these form a 1-cocycle in Čech cohomology with values in a group G . We develop here a 1-cohomology of a poset with values in G and regard this as describing principal bundles over spacetimes. A different 1-cohomology has already proved useful in algebraic quantum field theory: a cohomology of the poset with values in a net of observables describes the superselection sectors. The formalism developed here can be adapted to this case.

We begin by explaining the notions of simplex, path and homotopy in the context of posets showing that these notions behave in much the same way as their better known topological counterparts. We define the fundamental group of a path-connected poset which, in practice, coincides with the fundamental group of the spacetime. We then explain the 1-cohomology of a poset with values in G linking it to homotopy: the category of 1-cocycles is equivalent to the category of homomorphisms from the fundamental group to G .

Having defined principal bundles, we next introduce the appropriate notion of connection and curvature and investigate the set of connections on a principal bundle, these being thus associated with a particular 1-cohomology. We discuss holonomy and prove a version of the Ambrose-Singer Theorem.

We finally introduce the notion of gauge transformation and the action of the group of gauge transformations on the set of connections of a principal bundle. We also relate flat connections to homomorphisms from the fundamental group into G . We end by giving a brief outlook.

2 Homotopy of posets

We introduce some preliminary notions and results on posets. We will start by defining the simplicial set associated with a poset and arrive at the notion of a simply connected poset. Throughout this section, we will consider a poset K and denote its order relation by \leq . References for this section are [10, 11, 14].

The simplicial set of K : Underlying cohomology is what is called the simplicial category Δ^+ that can be realized in various ways. The simplest way is to take the objects of Δ^+ to be the finite ordinals, $\Delta_n = \{0, 1, \dots, n\}$ and to take the arrows to be the monotone mappings. All these monotone mappings are compositions of two particular simple types of mapping; the injective monotone mappings from one ordinal to the succeeding ordinal denoted $d_i^n : (n-1) \rightarrow n$, with $i \in \{0, 1, \dots, n\}$, and defined as

$$d_i^n(k) \equiv \begin{cases} k & k < i, \\ k+1 & \text{otherwise}; \end{cases}$$

and the surjective monotone mappings from one ordinal to the preceding one denoted $s_i^n : (n+1) \rightarrow n$, with $i \in \{0, 1, \dots, n\}$, and defined as

$$s_i^n(k) \equiv \begin{cases} k & k \leq i, \\ k-1 & \text{otherwise}. \end{cases}$$

The superscripts of the symbols d_i^n and s_i^n are usually omitted. The following identities allow one to compute effectively

$$d_i d_j = d_{j+1} d_i, \quad i \leq j; \quad s_j s_i = s_i s_{j+1}, \quad i \leq j;$$

$$s_j d_i = d_i s_{j-1}, \quad i < j; \quad s_j d_j = s_j d_{j+1} = 1; \quad s_j d_i = d_{i-1} s_j, \quad i > j+1.$$

Actually, each monotone map can be factorized uniquely as the composition of a surjective monotone map and an injective monotone map.

We may also regard Δ_n as a partially ordered set, namely as the set of its non-void subsets ordered under inclusion. We denote this poset by $\tilde{\Delta}_n$. Any map, in particular a monotone one, $m : \Delta_n \rightarrow \Delta_p$ induces, in an obvious way, an order-preserving map of the partially ordered sets $\tilde{\Delta}_n$ and $\tilde{\Delta}_p$, denoted by \tilde{m} . We can then define a *singular n -simplex* of a poset K to be an order preserving map $f : \tilde{\Delta}_n \rightarrow K$. We denote the set of singular n -simplices by $\Sigma_n(K)$, and call the *simplicial set* of K the set $\Sigma_*(K)$ of all singular simplices. Note that a map $m : \Delta_n \rightarrow \Delta_p$ induces a map

$m^* : \Sigma_p(K) \rightarrow \Sigma_n(K)$, where $m^*(f) \equiv f \circ \tilde{m}$ with $f \in \Sigma_p(K)$. In particular, we have maps

$$\begin{aligned} \partial_i : \Sigma_n(K) &\rightarrow \Sigma_{n-1}(K), \text{ where } \partial_i \equiv d_i^*, \\ \sigma_i : \Sigma_n(K) &\rightarrow \Sigma_{n+1}(K), \text{ where } \sigma_i \equiv s_i^*, \end{aligned} \quad (1)$$

called *boundaries* and *degeneracies*, respectively. One can easily check the following relations

$$\begin{aligned} \partial_i \partial_j &= \partial_j \partial_{i+1}, \quad i \geq j; & \sigma_i \sigma_j &= \sigma_{j+1} \sigma_i, \quad i \leq j; \\ \partial_i \sigma_j &= \sigma_{j-1} \partial_i, \quad i < j; & \partial_j \sigma_j &= \partial_{j+1} \sigma_j = 1; & \partial_i \sigma_j &= \sigma_j \partial_{i-1}, \quad i > j+1 \end{aligned} \quad (2)$$

From now on, we will denote: the composition $\partial_i \partial_j$ by the symbol ∂_{ij} ; 0-simplices by the letter a ; 1-simplices by b ; 2-simplices by c and a generic n -simplex by d . A 0-simplex a is just an element of the poset. Inductively, for $n \geq 1$, an n -simplex d is formed by $n+1$ $(n-1)$ -simplices $\partial_0 d, \dots, \partial_n d$, whose boundaries are constrained by the relations (2), and by a 0-simplex $|d|$ called the *support* of d such that $|\partial_0 d|, \dots, |\partial_n d| \leq |d|$. The ordered set $(\partial_0 d, \dots, \partial_n d)$, denoted ∂d , is called the *boundary* of d . We say that an n -simplex is *degenerate* if it is of the form $\sigma_i(d)$ for some $(n-1)$ -simplex d and for some $i \in \{0, 1, \dots, n-1\}$. For instance, by using the relations (2), it is easily seen that

$$\partial_0 \sigma_0(a) = \partial_1 \sigma_0(a) = a, \quad |\sigma_0(a)| = a,$$

for any 0-simplex a . In general, we have $|\sigma_i(d)| = |d|$.

In later applications the following class of simplices will be important. A 1-simplex b is said to be *inflating* whenever

$$\partial_1 b \subseteq \partial_0 b. \quad (3)$$

By induction for $n \geq 1$: an n -simplex d is said to be *inflating* whenever all its $(n-1)$ -boundaries $\partial_0 d, \dots, \partial_n d$ are inflating $(n-1)$ -simplices. For instance, if c is an inflating 2-simplex, then

$$\partial_{11} c = \partial_{12} c \subseteq \partial_{02} c = \partial_{10} c \subseteq \partial_{00} c = \partial_{01} c.$$

Any 0-simplex will be regarded as inflating. We will denote the set of inflating n -simplices by $\Sigma_n^{\text{inf}}(K)$. Given a monotone mapping $m : \Delta_p \rightarrow \Delta_n$ then $m^*(d) = d \circ \tilde{m}$ is inflating if d is. Thus $\Sigma_*^{\text{inf}}(K)$ is a simplicial subset of $\Sigma_*(K)$.

We now deal with the notion of orientation of singular simplices of a poset K . In general, one says that a pair of simplices have the same orientation whenever one can be obtained from the other by means of an even permutation of its vertices. The resulting equivalence relation will be called *oriented equivalence*. Notice that the k^{th} -vertex associated with an n -simplex d is the 0-simplex given by $\partial_{012\ldots\widehat{k}\ldots n}d$, where \widehat{k} means that the index k is omitted. Given a permutation σ of $(n+1)$ elements we denote by d^σ the n -simplex obtained by permuting the vertices of d according to σ and leaving fixed the related supports. To be precise, we define d^σ as the n -simplex such that $|d^\sigma| = |d|$ and

$$\partial_{012\ldots\widehat{k}\ldots n}d^\sigma = \partial_{012\ldots\widehat{\sigma(k)}\ldots n}d, \quad k \in \{0, 1, \ldots, n\} \quad (4)$$

These $n+1$ relations and the commutation relations (2) allow one to compute how the boundaries of d^σ are related to those of d . As an example, let σ be the transposition (01). Then $a^{(01)} = a$ for any 0-simplex a . Inductively for $n \geq 1$, if d is an n -simplex, then $|d^{(01)}| = |d|$ and

$$\partial_0 d^{(01)} = \partial_1 d, \quad \partial_1 d^{(01)} = \partial_0 d, \quad \partial_i d^{(01)} = (\partial_i d)^{(01)} \quad i \in \{2, 3, \ldots, n\}. \quad (5)$$

Now, observe that the mapping

$$\mathbb{P}(n+1) \times \Sigma_n(K) \ni (\sigma, d) \rightarrow d^\sigma \in \Sigma_n(K),$$

where $\mathbb{P}(n+1)$ is the group of the permutations of $(n+1)$ elements, defines an action of $\mathbb{P}(n+1)$ on $\Sigma_n(K)$. Two n -simplices d and d_1 are said to have the *same orientation* if there exists an *even* permutation σ of $\mathbb{P}(n+1)$ such that $d_1 = d^\sigma$; they have a *reverse orientation* if there is an *odd* permutation σ of $\mathbb{P}(n+1)$ such that $d_1 = d^\sigma$. We denote by $[d]$ the equivalence class of 1-simplices which have the same orientation as d , and by $\overline{[d]}$ the equivalence class of 1-simplices whose orientation is the reverse of d . Notice that for any 0-simplex a we have $[a] = \overline{[a]} = a$. For 1-simplices we have $[b] = \{b\}$, while $\overline{[b]} = \{\overline{b}\}$, where \overline{b} is the 1-simplex defined as

$$|\overline{b}| = |b|, \quad \partial_0 \overline{b} = \partial_1 b, \quad \partial_1 \overline{b} = \partial_0 b. \quad (6)$$

In the following we will refer to \overline{b} as the *reverse* of the 1-simplex b (note that $\overline{\overline{b}} = b^{(01)}$). For a 2-simplex c we have

$$[c] = \{c, c^{(02)(01)}, c^{(12)(01)}\}, \quad \overline{[c]} = \{c^{(01)}, c^{(02)}, c^{(12)}\}. \quad (7)$$

For instance, $|c^{(02)(01)}| = |c|$ and

$$\partial_0 c^{(02)(01)} = \partial_2 c, \quad \partial_1 c^{(02)(01)} = \overline{\partial_0 c}, \quad \partial_2 c^{(02)(01)} = \overline{\partial_1 c}; \quad (8)$$

while $|c^{(12)(01)}| = |c|$ and

$$\partial_0 c^{(12)(01)} = \overline{\partial_1 c}, \quad \partial_1 c^{(12)(01)} = \overline{\partial_2 c}, \quad \partial_2 c^{(12)(01)} = \partial_0 c. \quad (9)$$

In contrast to the usual cohomological theories, we do not identify an n -simplex d with its equivalence class $[d]$. This is because in the following we will deal with the curvature of a connection which is, in general, not invariant under oriented equivalence.

Paths : Given $a_0, a_1 \in \Sigma_0(K)$, a *path* from a_0 to a_1 is a finite ordered sequence $p = \{b_n, \dots, b_1\}$ of 1-simplices satisfying the relations

$$\partial_1 b_1 = a_0, \quad \partial_0 b_i = \partial_1 b_{i+1} \quad \text{for } i \in \{1, \dots, n-1\}, \quad \partial_0 b_n = a_1.$$

The *starting point* of p , written $\partial_1 p$, is the 0-simplex a_0 , while the *endpoint* of p , written $\partial_0 p$, is the 0-simplex a_1 . The *boundary* of p is the ordered set $\partial p \equiv \{\partial_0 p, \partial_1 p\}$. A path p is said to be a *loop* if $\partial_0 p = \partial_1 p$. The *support* $|p|$ of the path p is the set

$$|p| \equiv \{|b_1|, \dots, |b_n|\}.$$

We will denote the set of paths from a_0 to a_1 by $K(a_0, a_1)$, and the loops having endpoint a_0 by $K(a_0)$. K will be assumed to be *pathwise connected*, i.e. $K(a_0, a_1)$ is never void. The set of paths is equipped with the following operations. Consider a path $p = \{b_n, \dots, b_1\} \in K(a_0, a_1)$. The *reverse* \bar{p} is the path

$$\bar{p} \equiv \{\bar{b}_1, \dots, \bar{b}_n\} \in K(a_1, a_0).$$

The *composition* of p with a path $q = \{b'_k, \dots, b'_1\}$ of $K(a_1, a_2)$, is defined by

$$q * p \equiv \{b'_k, \dots, b'_1, b_n, \dots, b_1\} \in K(a_0, a_2).$$

Note that the reverse $-$ is involutive and the composition $*$ is associative. In particular note that any path $p = \{b_n, \dots, b_1\}$ can be also seen as the composition of its 1-simplices, i.e., $p = b_n * \dots * b_1$.

An *elementary deformation* of a path p consists in replacing a 1-simplex $\partial_1 c$ of the path by a pair $\partial_0 c, \partial_2 c$, where $c \in \Sigma_2(K)$, or, conversely in replacing a consecutive pair $\partial_0 c, \partial_2 c$ of 1-simplices of p by a single 1-simplex $\partial_1 c$. Two paths with the same endpoints are *homotopic* if they can be obtained from one other by a finite set of elementary deformations. Homotopy defines an equivalence relation \sim on the set of paths with the same endpoints, which is compatible with reverse and composition, namely

$$\begin{aligned} p \sim q &\iff \bar{p} \sim \bar{q}, & p, q \in K(a_0, a_1); \\ p \sim q, \quad p_1 \sim q_1 &\implies p_1 * p \sim q_1 * q, & p_1, q_1 \in K(a_1, a_2). \end{aligned} \quad (10)$$

Furthermore, for any $p \in K(a_0, a_1)$, the following relations hold:

$$\begin{aligned} p * \sigma_0(a_0) &\sim p & \text{and} & & p &\sim \sigma_0(a_1) * p; \\ \bar{p} * p &\sim \sigma_0(a_0) & \text{and} & & \sigma_0(a_1) &\sim p * \bar{p}, \end{aligned} \quad (11)$$

where $\sigma_0(a_0)$ is the 1-simplex degenerate to a_0 .

The first homotopy group: Fix $a_0 \in \Sigma_0(K)$, and define

$$\pi_1(K, a_0) \equiv K(a_0) / \sim,$$

the quotient of the set of loops with endpoints a_0 by the homotopy equivalence relation. Let $[p]$ be the equivalence class associated with the loop $p \in K(a_0)$, and let

$$[p] \cdot [q] = [p * q], \quad [p], [q] \in \pi_1(K, a_0).$$

$\pi_1(K, a_0)$ with this composition rule is a group: the identity is the equivalence class $[\sigma_0(a_0)]$ associated with the degenerate 1-simplex $\sigma_0(a_0)$; the inverse of $[p]$ is the equivalence class $[\bar{p}]$ associated with the reverse \bar{p} of p . $\pi_1(K, a_0)$ is the *first homotopy group* of K based on a_0 . Since K is pathwise connected $\pi_1(K, a_0)$ is isomorphic to $\pi_1(K, a)$ for any $a \in \Sigma_0(K)$; this isomorphism class is the *fundamental group* of K , written $\pi_1(K)$. If $\pi_1(K)$ is trivial, then K is said to be *simply connected*. It turns out that if K is directed¹, then K is simply connected.

The link between the homotopy group of a poset and the corresponding topological notion, can be achieved as follows. Let M be an arcwise connected manifold and let K be a base for the topology of M whose elements are arcwise and simply connected, open subsets of M . Consider the poset formed by ordering K under *inclusion*. Then $\pi_1(M) = \pi_1(K)$, where $\pi_1(M)$ is the fundamental group of M .

Coverings : A partially ordered set K can be equipped with a T_0 topology called the Alexandroff topology. In this topology, a subset $U \subseteq K$ is said to be *open* whenever given $\mathcal{O} \in U$ and $\mathcal{O}_1 \in K$, if $\mathcal{O} \leq \mathcal{O}_1$ then $\mathcal{O}_1 \in U$. An *open covering* of K is a family \mathcal{U} of open sets U of K such that for any $\mathcal{O} \in K$ there is $U \in \mathcal{U}$ with $\mathcal{O} \in U$. A particular covering is that formed by the collection $\{U_a, a \in \Sigma_0(K)\}$ of open sets of K defined by

$$U_a \equiv \{\mathcal{O} \in K \mid a \leq \mathcal{O}\}, \quad a \in \Sigma_0(K). \quad (12)$$

¹The poset K is directed whenever for any pair $\mathcal{O}, \mathcal{O}_1 \in K$ there exists $\mathcal{O}_2 \in K$ such that $\mathcal{O}, \mathcal{O}_1 \leq \mathcal{O}_2$

We call this covering the *fundamental covering* of K . Note that if \mathcal{U} is an open covering of K , then for any 0-simplex a there is $U \in \mathcal{U}$ such that $U_a \subseteq U$.

3 Cohomology of posets

The present section deals with the, in general non-Abelian, cohomology of a pathwise connected poset K with values in a group G . The first part is devoted to explaining the motivation for studying the non-Abelian cohomology of a poset and to defining an n -category. The general theory is developed in the second part: we introduce the set of n -cochains, for $n = 0, 1, 2, 3$, the coboundary operator, and the cocycle identities up to the 2^{nd} -degree. In the last part we study the 1-cohomology, in some detail, relating it to the first homotopy group of a poset.

3.1 Preliminaries

The cohomology of the poset K with values in an Abelian group A , written additively, is the cohomology of the set of singular simplices $\Sigma_*(K)$ with values in A . To be precise, one can define the set $C^n(K, A)$ of n -cochains of K with values in A as the set of functions $v : \Sigma_n(K) \rightarrow A$. The coboundary operator d defined by

$$dv(d) = \sum_{k=0}^n (-1)^k v(\partial_k d), \quad d \in \Sigma_n(K),$$

is a mapping $d : C^n(K, A) \rightarrow C^{n+1}(K, A)$ satisfying the equation $ddv = \iota$, where ι is the trivial cochain. This allows one to define the n -cohomology groups. For a non-Abelian group G no choice of ordering gives the identity $ddv = \iota$.

One motivation for studying the cohomology of a poset K with values in a non-Abelian group comes from algebraic quantum field theory. The leading idea of this approach is that all the physical content of a quantum system is encoded in the observable net, an inclusion preserving correspondence which associates to any open and bounded region of Minkowski space the algebra generated by the observables measurable within that region. The collection of these regions forms a poset when ordered under inclusion. A 1-cocycle equation arises in studying charged sectors of the observable net: the charge transporters of sharply localized charges are 1-cocycles of the poset taking values in the group of unitary operators of the observable net [8]. The

attempt to include more general charges in the framework of local quantum physics, charges of electromagnetic type in particular, has led one to derive higher cocycles equations, up to the third degree [9, 10]. The difference, with respect to the Abelian case, is that a n -cocycle equation needs n -composition laws. Thus in non-Abelian cohomology instead, for example, of trying to take coefficients in a non-Abelian group the n -cocycles take values in an n -category associated with the group. The cocycles equations can be understood as pasting together simplices, and, in fact, a n -cocycle can be seen as a representation in an n -category of the algebra of an oriented n -simplex [15].

Before trying to learn the notion of an n -category, it helps to recall that a category can be defined in two equivalent manners. One definition is based on the set of objects and the corresponding set of arrows. However, it is possible to define a category referring only to the set of arrows. Namely, a category is a set \mathcal{C} , whose elements are called arrows, having a partial and associative composition law \diamond , and such that any element of \mathcal{C} has left and right \diamond -units. This amounts to saying that (i) $(f \diamond g) \diamond h$ is defined if, and only if, $f \diamond (g \diamond h)$ is defined and they are equal; (ii) the triple $f \diamond g \diamond h$ is defined if, and only if, $f \diamond g$ and $g \diamond h$ are defined; (iii) any arrow g has a left and a right unit u and v , that is $u \diamond g = g$ and $g \diamond v = g$. In this formulation the set of objects are the set of units.

An n -category is a set \mathcal{C} with an ordered set of n partial composition laws. This means that \mathcal{C} is a category with respect to any such composition law \diamond . Moreover, if \times and \diamond are two such composition laws with $\times \prec \diamond$ then:

1. every \times -unit is a \diamond -unit;
2. \times -composition of \diamond -units, when defined, leads to \diamond -units;
3. the following relation, called the *interchange law*, holds:

$$(f \times h) \diamond (f_1 \times h_1) = (f \diamond f_1) \times (h \diamond h_1),$$

whenever the right hand side is defined.

An arrow f is said to be a k -arrow, for $k \leq n$, if it is a unit for the $k + 1$ composition law. To economize on brackets, from now on we adopt the convention that if $\times \prec \diamond$, then a \times -composition law is to be evaluated before a \diamond -composition. For example, the interchange law reads

$$f \times h \diamond f_1 \times h_1 = (f \diamond f_1) \times (h \diamond h_1).$$

It is surprising that with this convention all the brackets disappear from the coboundary equations (see below).

That an n -category is the right set of coefficients for a non-Abelian cohomology can be understood by the following observation. Assume that \times is Abelian, that is, $f \times g$ equals $g \times f$ whenever the compositions are defined. Assume that \diamond -units are \times -units. Let $1, 1'$ be, respectively, a left and a right \diamond -unit for f and g . By using the interchange law we have

$$f \diamond g = 1 \times f \diamond g \times 1' = (1 \diamond g) \times (f \diamond 1') = g \times f.$$

Hence \diamond equals \times and both composition laws are Abelian. Furthermore, if \star is a another composition law such that $\times \prec \star \prec \diamond$, then $\times = \star = \diamond$.

3.2 Non-Abelian cohomology

The first aim is to introduce an n -category associated with a group G to be used as set of coefficients for the cohomology of the poset K . To this end, we draw on a general procedure [12] associating to any n -category \mathcal{C} where the n -arrows are invertible, with respect to any composition law, an $(n+1)$ -category $\mathcal{I}(\mathcal{C})$ with the same property. This construction allows one to define the $(n+1)$ -coboundary of a n -cochain in \mathcal{C} as an $(n+1)$ -cochain in $\mathcal{I}(\mathcal{C})$, at least for $n = 0, 1, 2$.

Before starting to describe non-Abelian cohomology, we introduce some notation. The elements of a group G will be indicated by Latin letters. The composition of two elements g, h of G will be denoted by gh , and by e the identity of G . Let $\text{Inn}(G)$ be the group of inner automorphisms of G . We will use Greek letters to indicate the elements of $\text{Inn}(G)$. By $\alpha\tau$ we will denote the inner automorphism of G obtained by the composing α with τ , that is $\alpha\tau(h) \equiv \alpha(\tau(h))$ for any $h \in G$. The identity of this group, the trivial automorphism, will be indicated by ι . Finally given $g \in G$, the equation

$$g\alpha = \tau g$$

means $g\alpha(h) = \tau(h)g$ for any $h \in G$.

The categories nG : In degree 0, this is simply the group G considered as a set. In degree 1 it is the category $1G$ having a single object, the group G , and as arrows the elements of the group. Composition of arrows is the composition in G . So we identify this category with G . Observe that the arrows of $1G$ are invertible. By applying the procedure provided in [12] we

have that $\mathcal{I}(1G)$ is a 2-category, denoted by $2G$, whose set of arrows is

$$2G \equiv \{(g, \tau) \mid g \in G, \tau \in \text{Inn}(G)\}, \quad (13)$$

and whose composition laws are defined by

$$\begin{aligned} (g, \tau) \times (h, \gamma) &\equiv (g\tau(h), \tau\gamma), \\ (g, \tau) \diamond (h, \gamma) &\equiv (gh, \gamma), \quad \text{if } \sigma_h\gamma = \tau, \end{aligned} \quad (14)$$

where σ_h is the inner automorphism associated with h . Some observations on $2G$ are in order. Note that the composition \times is always defined. Furthermore, the set of 1-arrows is the set of those elements of $2G$ of the form (e, τ) . Finally, all the 2-arrows are invertible. We can now construct the 3-category $\mathcal{I}(2G)$, denoted by $3G$. It turns out that $3G$ is the set

$$3G \equiv \{(g, \tau, \gamma) \mid g \in \mathcal{Z}(G), \tau, \gamma \in \text{Inn}(G)\}, \quad (15)$$

where $\mathcal{Z}(G)$ is the centre of G , with the following three composition laws

$$\begin{aligned} (g, \tau, \gamma) \times (g', \tau', \gamma') &\equiv (gg', \tau\tau', \gamma\tau\gamma'\tau^{-1}), \\ (g, \tau, \gamma) \diamond (g', \tau', \gamma') &\equiv (gg', \tau', \gamma\gamma'), \quad \text{if } \tau = \gamma'\tau' \\ (g, \tau, \gamma) \cdot (g', \tau', \gamma') &\equiv (gg', \tau, \gamma), \quad \text{if } \tau = \tau', \gamma = \gamma'. \end{aligned} \quad (16)$$

Note that \cdot is Abelian. The set of 1-arrows $(3G)_1$ is the subset of elements of $3G$ of the form (e, γ, ι) , where ι denotes the identity automorphism; 2-arrows $(3G)_2$ are the elements of $3G$ of the form (e, τ, γ) . Finally, if G is Abelian, then $\times = \diamond = \cdot$ and the categories $2G$ and $3G$ are nothing but that the group G .

The set $\Sigma_n(K, G)$ of n -cochains : The next goal is to define the set of n -cochains. Concerning 0- and 1-cochains nothing change with respect to the Abelian case, i.e., 0- and 1-cochains are, respectively functions $v : \Sigma_0(K) \rightarrow G$ and $u : \Sigma_1(K) \rightarrow G$. A 2-cochain w is a pair of mappings (w_1, w_2) , where $w_i : \Sigma_i(K) \rightarrow (2G)_i$, for $i = 1, 2$ enjoying the relation

$$w_2(c) \diamond w_1(\partial_1 c) = w_1(\partial_0 c) \times w_1(\partial_2 c) \diamond w_2(c), \quad c \in \Sigma_2(K). \quad (17)$$

This equation and the definition of the composition laws in $2G$ entail that a 2-cochain w is of the form

$$\begin{aligned} w_1(b) &= (e, \tau_b), \quad b \in \Sigma_1(K), \\ w_2(c) &= (v(c), \tau_{\partial_1 c}), \quad c \in \Sigma_2(K), \end{aligned} \quad (18)$$

where $v : \Sigma_2(K) \rightarrow G$, $\tau : \Sigma_1(K) \rightarrow \text{Inn}(G)$ are mapping satisfying the equation²

$$v(c) \tau_{\partial_1 c} = \tau_{\partial_0 c} \tau_{\partial_2 c} v(c), \quad c \in \Sigma_2(K). \quad (19)$$

This can be easily shown. In fact, according to the definition of $2G$ a 2-cochain w is of the form $w_1(b) = (e, \tau_b)$ for $b \in \Sigma_1(K)$, and $w_2(c) = (v(c), \beta_c)$ for $c \in \Sigma_2(K)$. Now, the l.h.s. of equation (17) is defined if, and only if, $\tau_{\partial_1 c} = \beta_c$ for any 2-simplex c . This fact and equation (17) entail (19) and (18), completing the proof.

A 3-cochain x is 3-tuple (x_1, x_2, x_3) where $x_i : \Sigma_1(K) \rightarrow (3G)_i$, for $i = 1, 2, 3$, satisfying the following equations

$$x_2(c) \diamond x_1(\partial_1 c) = x_1(\partial_0 c) \times x_1(\partial_2 c) \diamond x_2(c), \quad (20)$$

for any 2-simplex c , and

$$\begin{aligned} x_3(d) \cdot x_1(\partial_{01} d) \times x_2(\partial_3 d) \diamond x_2(\partial_1 d) = \\ = x_2(\partial_0 d) \times x_1(\partial_{23} d) \diamond x_2(\partial_2 d) \cdot x_3(d), \end{aligned} \quad (21)$$

for any 3-simplex d . Proceeding as above, these equations and the composition laws of $3G$ entail that a 3-cochain x is of the form

$$\begin{aligned} x_1(b) &= (e, \tau_b, \iota), & b \in \Sigma_1(K), \\ x_2(c) &= (e, \tau_{\partial_1 c}, \gamma_c), & c \in \Sigma_2(K), \\ x_3(d) &= (v(d), \tau_{\partial_{12} d}, \gamma_{\partial_0 d} \gamma_{\partial_1 d}), & d \in \Sigma_3(K), \end{aligned} \quad (22)$$

where $\tau : \Sigma_1(K) \rightarrow \text{Inn}(G)$, $v : \Sigma_3(K) \rightarrow Z(G)$, while $\gamma : \Sigma_2(K) \rightarrow \text{Inn}(G)$ is the mapping defined as

$$\gamma_c \equiv \tau_{\partial_0 c} \tau_{\partial_2 c} \tau_{\partial_1 c}^{-1}, \quad c \in \Sigma_2(K). \quad (23)$$

Note, in particular that $\gamma_c \tau_{\partial_1 c} = \tau_{\partial_0 c} \tau_{\partial_2 c}$ for any 2-simplex c . This concludes the definition of the set of cochains. We will denote the set of n -cochains of K , for $n = 0, 1, 2, 3$, by $C^n(K, G)$.

Just a comment about the definition of 1-cochains. Unlike the usual cohomological theories 1-cochains are neither required to be invariant under oriented equivalence of simplices nor to act trivially on degenerate simplices. However, as we will see later, 1-cocycles and connections fulfil these properties.

²Equation 19 means that $v(c)$ intertwines $\tau_{\partial_1 c}$ with $\tau_{\partial_0 c} \tau_{\partial_2 c}$, that is $v(c) \tau_{\partial_1 c}(h) = \tau_{\partial_0 c}(\tau_{\partial_2 c}(h)) v(c)$ for any, $h \in G$.

The coboundary and the cocycle identities: The next goal is to define the coboundary operator d . Given a 0-cochain v , then

$$dv(b) \equiv v(\partial_0 b) v(\partial_1 b)^{-1}, \quad b \in \Sigma_1(K). \quad (24)$$

Given a 1-cochain u , then

$$\begin{aligned} (du)_1(b) &\equiv (e, \text{ad}(u(b))), & b \in \Sigma_1(K), \\ (du)_2(c) &\equiv (w_u(c), \text{ad}(u(\partial_1 c))), & c \in \Sigma_2(K), \end{aligned} \quad (25)$$

where w_u is the mapping from $\Sigma_2(K)$ to G defined as

$$w_u(c) \equiv u(\partial_0 c) u(\partial_2 c) u(\partial_1 c)^{-1}, \quad c \in \Sigma_2(K). \quad (26)$$

Finally, given a 2-cochain w of the form (18), then

$$\begin{aligned} (dw)_1(b) &\equiv (e, \tau_b, \iota), & b \in \Sigma_1(K), \\ (dw)_2(c) &\equiv (e, \tau_{\partial_1 c}, \gamma_c), & c \in \Sigma_2(K), \\ (dw)_3(d) &\equiv (x_w(d), \tau_{\partial_{12}d}, \gamma_{\partial_0 d} \gamma_{\partial_2 d}), & d \in \Sigma_3(K), \end{aligned} \quad (27)$$

where γ is the function from $\Sigma_2(K)$ to $\text{Inn}(G)$ defined by τ as in (23), and x_w is the mapping $x_w : \Sigma_3(K) \rightarrow \mathcal{Z}(G)$ defined as

$$x_w(d) \equiv v(\partial_0 d) v(\partial_2 d) (\tau_{\partial_{01}d} (v(\partial_3 d)) v(\partial_1 d))^{-1} \quad (28)$$

for any 3-simplex d . Now, we call the *coboundary operator* d the mapping $d : C^n(K, G) \rightarrow C^{n+1}(K, G)$ defined for $n = 0, 1, 2$ by the equations (24), (25) and (27) respectively. This definition is well posed as shown by the following

Lemma 3.1. *For $n = 0, 1, 2$, the coboundary operator d is a mapping $d : C^n(K, G) \rightarrow C^{n+1}(K, G)$, such that*

$$ddv \in ((2+k)G)_{k+1}, \quad v \in C^k(K, G)$$

for $k = 0, 1$.

Proof. The proof of the first part of the statement follows easily from the definition of d , except that the function x_w , as defined in (28), takes values in $\mathcal{Z}(G)$. Writing, for brevity, v_i for $v(\partial_i d)$ and τ_{ij} for $\tau_{\partial_{ij}}$, and using relations (2) and equation (19) we have

$$v_0 v_2 \tau_{12} = v_0 \tau_{02} \tau_{22} v_2 = v_0 \tau_{10} \tau_{22} v_2 = \tau_{00} \tau_{20} \tau_{23} v_0 v_2,$$

moreover

$$\begin{aligned}
\tau_{01}(v_3) v_1 \tau_{12} &= \tau_{01}(v_3) v_1 \tau_{11} = \tau_{01}(v_3 d) \tau_{01} \tau_{21} v_1 \\
&= \tau_{01}(v_3 d \tau_{13}) v_1 = \tau_{01}(\tau_{03} \tau_{23} v_3) v_1 \\
&= \tau_{01} \tau_{03} \tau_{23} \tau_{01}(v_3) v_1,
\end{aligned}$$

Hence both $v(\partial_0 d) v(\partial_2 d)$ and $\tau_{\partial_0 d}(v(\partial_3 d)) v(\partial_1 d)$ intertwine from $\tau_{\partial_{12} d}$ to $\tau_{\partial_0 d} \tau_{\partial_{03} d} \tau_{\partial_{23} d}$. This entails that they differ only by an element of $\mathcal{Z}(G)$, and this proves that x_w takes values in $\mathcal{Z}(G)$. Now, it is very easy to see that $\text{dd}v \in (2G)_1$, for any 0-cochain v . So, let us prove that $\text{dd}u \in (3G)_2$ for any 1-cochain u . Note that

$$\begin{aligned}
(\text{du})_1(b) &= (e, \text{ad}(u(b))), & b \in \Sigma_1(K) \\
(\text{du})_2(c) &= (w_u(c), \text{ad}(u(\partial_1 c))), & c \in \Sigma_2(K),
\end{aligned}$$

where w_u is defined by (26). Then the proof follows once we have shown that

$$w_u(\partial_0 d) w_u(\partial_2 d) = \text{ad}(u(\partial_0 d))(w_u(\partial_3 d)) w_u(\partial_1 d), \quad (*)$$

for any 3-simplex d . In fact, by (27) this identity entails that

$$\begin{aligned}
(\text{dd}u)_1(b) &= (e, \text{ad}(u(b)), \iota), & b \in \Sigma_1(K) \\
(\text{dd}u)_2(c) &= (e, \text{ad}(u(\partial_1 c)), \text{ad}(w_u(c))), & c \in \Sigma_2(K), \\
(\text{dd}u)_3(d) &= (e, \text{ad}(u(\partial_{12} d)), \text{ad}(w_u(\partial_0 d) w_u(\partial_2 d))), & d \in \Sigma_3(K).
\end{aligned}$$

So let us prove (*). Given $d \in \Sigma_3(K)$ and using relations (2), we have

$$\begin{aligned}
\text{ad}(u(\partial_0 d))(w_u(\partial_3 d)) w_u(\partial_1 d) &= \\
&= u(\partial_0 d) w_u(\partial_3 d) u(\partial_0 d)^{-1} w_u(\partial_1 d) \\
&= u_{01} (u_{03} u_{23} u_{13}^{-1}) u_{01}^{-1} (u_{01} u_{21} u_{11}^{-1}) \\
&= u_{01} u_{03} u_{23} u_{11}^{-1} \\
&= u_{01} u_{03} u_{02}^{-1} u_{02} u_{23} u_{11}^{-1} \\
&= u_{00} u_{20} u_{10}^{-1} u_{02} u_{22} u_{12}^{-1} \\
&= w_u(\partial_0 d) w_u(\partial_2 d),
\end{aligned}$$

where we have used the notation introduced above. This completes the proof. \square

In words this lemma says that if v is a 0-cochain, then ddv is a 2-unit of $2G$; if u is a 1-cochain, then ddv is a 3-unit of $3G$.

We now are in a position to introduce the definition of an n -cocycle.

Definition 3.2. For $n = 0, 1, 2$, an n -cochain v is said to be an n -**cocycle** whenever

$$dv \in ((n+1)G)_n.$$

It is said to be an n -**coboundary** whenever

$$v \in d((n-1)G)$$

(for $n = 0$ this means that $v(a) = e$ for any 0-simplex a). We will denote the set of n -cocycles by $Z^n(K, G)$, and by $B^n(K, G)$ the set of n -coboundaries.

Lemma 3.1 entails that $B^n(K, G) \subseteq Z^n(K, G)$ for $n = 0, 1, 2$. Although it is outside the scope of this paper, we note that this relation holds also for $n = 3$. One can check this assertion by using the 3-cocycle given in [9].

It is very easy now to derive the cocycle equations. A 0-cochain v is a 0-cocycle if

$$v(\partial_0 b) = v(\partial_1 b), \quad b \in \Sigma_1(K). \quad (29)$$

A 1-cochain z is a 1-cocycle if

$$z(\partial_0 c) z(\partial_2 c) = z(\partial_1 c), \quad c \in \Sigma_2(K). \quad (30)$$

Let $w = (w_1, w_2)$ be 2-cochain of the form $w_1(b) = (e, \tau_b)$ for $b \in \Sigma_1(K)$, $w_2(c) = (v(c), \tau_{\partial_1 c})$ for $c \in \Sigma_2(K)$, where v and τ are mappings satisfying (19). Then w is a 2-cocycle if

$$v(\partial_0 d) v(\partial_2 d) = \tau_{\partial_0 d}(v(\partial_3 d)) v(\partial_1 d), \quad d \in \Sigma_3(K). \quad (31)$$

In the following we will mainly deal with 1-cohomology. Our purpose will be to show that the notion of 1-cocycle admits an interpretation as a principal bundle over a poset and that this kind of bundle admits connections. The 2-coboundaries enter the game as the curvature of connections. Since the poset is pathwise connected, it turns out that any 0-cocycle v is a constant function. Thus the 0-cohomology of K yields no useful information.

3.3 1-Cohomology

This section is concerned with 1-cocycles of the poset K . In the first part we introduce some basic notions that will be used throughout this paper. The

second part deals with 1-cocycles. We will derive some results confirming the interpretation of a 1-cocycle as a principal bundle over a poset. This interpretation will become clear in Section 4. In the last part we discuss the connection between 1-cohomology and homotopy of posets.

The category of 1-cochains : Given a 1-cochain $v \in C^1(K, G)$, we can and will extend v from 1-simplices to paths by defining for $p = \{b_n, \dots, b_1\}$

$$v(p) \equiv v(b_n) \cdots v(b_2) v(b_1). \quad (32)$$

Definition 3.3. Consider $v, v_1 \in C^1(K, G)$. A **morphism** f from v_1 to v is a function $f : \Sigma_0(K) \rightarrow G$ satisfying the equation

$$f_{\partial_0 p} v_1(p) = v(p) f_{\partial_1 p},$$

for all paths p . We denote the set of morphisms from v_1 to v by (v_1, v) .

There is an obvious composition law between morphisms given by point-wise multiplication and this makes $C^1(K, G)$ into a category. The identity arrow $1_v \in (v, v)$ takes the constant value e , the identity of the group. Given a group homomorphism $\gamma : G_1 \rightarrow G$ and a morphism $f \in (v_1, v)$ of 1-cochains with values in G_1 then $\gamma \circ v$, defined as

$$(\gamma \circ v)(b) \equiv \gamma(v(b)), \quad b \in \Sigma_1(K), \quad (33)$$

is a 1-cochain with values in G , and $\gamma \circ f$ defined as

$$(\gamma \circ f)_a \equiv \gamma(f_a), \quad a \in \Sigma_0(K), \quad (34)$$

is a morphism of $(\gamma \circ v_1, \gamma \circ v)$. One checks at once that $\gamma \circ$ is a functor from $C^1(K, G_1)$ to $C^1(K, G)$, and that if γ is a group isomorphism, then $\gamma \circ$ is an isomorphism of categories.

Note that $f \in (v_1, v)$ implies $f^{-1} \in (v, v_1)$, where f^{-1} here denotes the composition of f with the inverse of G . We say that v_1 and v are *equivalent*, written $v_1 \cong v$, whenever (v_1, v) is nonempty. Observe that a 1-cochain v is equivalent to the trivial 1-cochain ι if, and only if, it is a 1-coboundary. We will say that $v \in C^1(K, G)$ is *reducible* if there exists a proper subgroup $G_1 \subset G$ and a 1-cochain $v_1 \in C^1(K, G_1)$ with $\gamma \circ v_1$ equivalent to v , where γ denotes the inclusion $G_1 \subset G$. If v is not reducible it will be said to be *irreducible*.

A 1-cochain v is said to be *path-independent* whenever given a pair of paths p, q , then

$$\partial p = \partial q \Rightarrow v(p) = v(q). \quad (35)$$

Of course, if v is path-independent then so is any equivalent 1-cochain. It is worth observing that if γ is an injective homomorphism then v is path-independent if, and only if, $\gamma \circ v$ is path-independent.

Lemma 3.4. *Any 1-cochain is path-independent if, and only if, it is a 1-coboundary.*

Proof. Assume that $v \in C^1(K, G)$ is path-independent. Fix a 0-simplex a_0 . For any 0-simplex a , choose a path p_a from a_0 to a and define $f_a \equiv v(p_a)$. As v is path-independent, for any 1-simplex b we have

$$v(b) f_{\partial_1 b} = v(b) v(p_{\partial_1 b}) = v(b * p_{\partial_1 b}) = v(p_{\partial_0 b}) = f_{\partial_0 b}.$$

Hence v is a 1-coboundary, see 24. The converse is obvious. \square

1-Cocycles as principal bundles : Recall that a 1-cocycle $z \in Z^1(K, G)$ is a mapping $z : \Sigma_1(K) \rightarrow G$ satisfying the equation

$$z(\partial_0 c) z(\partial_2 c) = z(\partial_1 c), \quad c \in \Sigma_2(K)$$

Some observations are in order. First, the trivial 1-cochain ι is a 1-cocycle (see Section 3.1). So, from now on, we will refer to ι as the *trivial 1-cocycle*. Secondly, if z is a 1-cocycle then so is any equivalent 1-cochain. In fact, let $v \in C^1(K, G)$ and let $f \in (v, z)$. Given a 2-simplex c we have

$$\begin{aligned} v(\partial_0 c) v(\partial_2 c) &= f_{\partial_0 c}^{-1} z(\partial_0 c) f_{\partial_{10} c} f_{\partial_{02} c}^{-1} z(\partial_2 c) f_{\partial_{12} c} \\ &= f_{\partial_0 c}^{-1} z(\partial_0 c) z(\partial_2 c) f_{\partial_{12} c} = f_{\partial_0 c}^{-1} z(\partial_1 c) f_{\partial_{12} c} \\ &= f_{\partial_{01} c}^{-1} z(\partial_1 c) f_{\partial_{11} c} = v(\partial_1 c), \end{aligned}$$

where relations (2) have been used.

Lemma 3.5. *Let $\gamma : G_1 \rightarrow G$ be a group homomorphism. Given $v \in C^1(K, G_1)$ consider $\gamma \circ v \in C^1(K, G)$. Then: if v is a 1-cocycle, then $\gamma \circ v$ is a 1-cocycle; the converse holds if γ is injective.*

Proof. If v is a 1-cocycle, it is easy to see that $\gamma \circ v$ is a 1-cocycle too. Conversely, assume that γ is injective and that $\gamma \circ v$ is a 1-cocycle, then

$$\gamma(v(\partial_0 c) v(\partial_2 c)) = \gamma \circ v(\partial_0 c) \gamma \circ v(\partial_2 c) = \gamma \circ v(\partial_1 c) = \gamma(v(\partial_1 c))$$

for any 2-simplex c . Since γ is injective, v is a 1-cocycle. \square

Given a 1-cocycle $z \in Z^1(K, G)$, a *cross section* of z is a function $s : \Sigma_0(U) \rightarrow G$, where U is an open set of K , such that

$$z(b) s_{\partial_1 b} = s_{\partial_0 b}, \quad b \in \Sigma_1(U). \quad (36)$$

The cross section s is said to be *global* whenever $U = K$. A reason for the terminology cross section of a 1-cocycle is provided by the following

Lemma 3.6. *A 1-cocycle is a 1-coboundary if, and only if, it admits a global cross section.*

Proof. The proof follows straightforwardly from the definition of a global cross section and from the definition of a 1-coboundary. \square

Remark 3.7. Given a group G , it is very easy to define 1-coboundaries of the poset K with values in G . It is enough to assign an element $s_a \in G$ to any 0-simplex a and set

$$z(b) \equiv s_{\partial_0 b} s_{\partial_1 b}^{-1}, \quad b \in \Sigma_1(K).$$

It is clear that z is a 1-cocycle. It is a 1-coboundary because the function $s : \Sigma_0(K) \rightarrow G$ is a global cross section of z . As we shall see in the next section, the existence of 1-cocycles, which are not 1-coboundaries, with values in a group G is equivalent to the existence of nontrivial group homomorphisms from the first homotopy group of K into G .

We call *the category of 1-cocycles* with values in G , the full subcategory of $C^1(K, G)$ whose set of objects is $Z^1(K, G)$. We denote this category by the same symbol $Z^1(K, G)$ as used to denote the corresponding set of objects. It is worth observing that, given a group homomorphism $\gamma : G_1 \rightarrow G$, by Lemma 3.5, the restriction of the functor $\gamma \circ$ to $Z^1(K, G_1)$ defines a functor from $Z^1(K, G_1)$ into $Z^1(K, G)$.

We interpret 1-cocycles of $Z^1(K, G)$ as principal bundles over the poset K , having G as a structure group. It is very easy to see which notion corresponds to that of an associated bundle in this framework. Assume that there is an action $\alpha : G \times X \ni (g, x) \rightarrow \alpha(g, x) \in X$ of G on a set X . Consider the group homomorphism $\tilde{\alpha} : G \ni g \rightarrow \tilde{\alpha}_g \in \text{Aut}(X)$ defined as

$$\tilde{\alpha}_g(x) \equiv \alpha(g, x), \quad x \in X,$$

for any $g \in G$. Given a 1-cocycle $z \in Z^1(K, G)$, we call the 1-cocycle

$$\tilde{\alpha} \circ z \in Z^1(K, \text{Aut}(X)), \quad (37)$$

associated with z , where $\tilde{\alpha} \circ$ is the functor, associated with the group homomorphism $\tilde{\alpha}$, from the category $Z^1(K, G)$ into $Z^1(K, \text{Aut}(X))$.

Homotopy and 1-cohomology : The relation between the homotopy and the 1-cohomology of K has been established in [14]. Here we reformulate this result in the language of categories. We begin by recalling some basic properties of 1-cocycles. First, any 1-cocycle $z \in Z^1(K, G)$ is *invariant under homotopy*. To be precise given a pair of paths p and q with the same endpoints, we have

$$p \sim q \Rightarrow z(p) = z(q). \quad (38)$$

Secondly, the following properties hold:

$$\begin{aligned} (a) \quad z(\overline{p}) &= z(p)^{-1}, \quad \text{for any path } p; \\ (b) \quad z(\sigma_0(a)) &= e, \quad \text{for any 0-simplex } a, \end{aligned} \quad (39)$$

Now in order to relate the homotopy of a poset to 1-cocycles, a preliminary definition is necessary.

Fix a group S . Given a group G we denote the set of group homomorphisms from S into G by $H(S, G)$. For any pair $\sigma, \sigma_1 \in H(S, G)$ a *morphism* from σ_1 to σ is an element h of G such that

$$h \sigma_1(g) = \sigma(g) h, \quad g \in S. \quad (40)$$

The set of morphisms from σ_1 to σ is denoted by (σ_1, σ) and there is an obvious composition rule between morphisms yielding a category again denoted by $H(S, G)$. Given a group homomorphism $\gamma : G_1 \rightarrow G$, there is a functor $\gamma \circ : H(S, G_1) \rightarrow H(S, G)$ defined as

$$\begin{aligned} \gamma \circ \sigma &\equiv \gamma \sigma & \sigma &\in H(S, G_1); \\ \gamma \circ h &\equiv \gamma(h) & h &\in (\sigma, \sigma_1), \sigma, \sigma_1 \in H(S, G_1). \end{aligned} \quad (41)$$

When γ is a group isomorphism, then $\gamma \circ$ is an isomorphism of categories, too. Similarly, let S_1 be a group and let $\rho : S_1 \rightarrow S$ be a group homomorphism. Then there is a functor $\circ \rho : H(S, G) \rightarrow H(S_1, G)$ defined by

$$\begin{aligned} \sigma \circ \rho &\equiv \sigma \rho & \sigma &\in H(S, G); \\ h \circ \rho &\equiv h & h &\in (\sigma, \sigma_1), \sigma, \sigma_1 \in H(S, G). \end{aligned} \quad (42)$$

When ρ is a group isomorphism, then $\circ \rho$ is an isomorphism of categories, too.

Now, fix a base 0-simplex a_0 and consider the category $H(\pi_1(K, a_0), G)$ associated with the first homotopy group of the poset. Then

Proposition 3.8. *Given a group G and any 0-simplex a_0 the categories $Z^1(K, G)$ and $H(\pi_1(K, a_0), G)$ are equivalent.*

Proof. Let us start by defining a functor from $Z^1(K, G)$ to $H(\pi_1(K, a_0), G)$. Given $z, z_1 \in Z^1(K, G)$ and $f \in (z_1, z)$, define

$$\begin{aligned} F(z)([p]) &\equiv z(p), \quad [p] \in \pi_1(K, a_0); \\ F(f) &\equiv f_{a_0}. \end{aligned}$$

$F(z)$ is well defined since 1-cocycles are homotopy invariant. Moreover, it is easy to see by (39) that $F(z)$ is a group homomorphism from $\pi_1(K, a_0)$ into G . Note that

$$f_{a_0} F(z_1)([p]) = f_{a_0} z_1(p) = z(p) f_{a_0} = F(z)([p]) f_{a_0},$$

hence $F(f) \in (F(z_1), F(z))$. So F is well defined and easily shown to be a covariant functor. To define a functor C in the other direction, let us choose a path p_a from a_0 to a , for any $a \in \Sigma_0(K)$. In particular we set $p_{a_0} = \sigma_0(a_0)$. Given $\sigma \in H(\pi_1(K, a_0), G)$ and $h \in (\sigma_1, \sigma)$, define

$$\begin{aligned} C(\sigma)(b) &\equiv \sigma(\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}), \quad b \in \Sigma_1(K); \\ C(h) &\equiv c(h), \end{aligned}$$

where $c(h) : \Sigma_0(K) \rightarrow G$ is the constant function taking the value h for any $a \in \Sigma_0(K)$. It can be easily shown that C is a covariant functor. Concerning the equivalence, note that

$$(F \cdot C)(\sigma)([p]) = C(\sigma)(p) = \sigma(\overline{[\sigma_0(a_0)]} * p * \sigma_0(a_0)) = \sigma([p]),$$

and that

$$(F \cdot C)(h) = F(c(h)) = h.$$

Hence $F \cdot C = \text{id}_{H(\pi_1(K, a_0), G)}$. Conversely, given a 1-simplex b we have

$$(C \cdot F)(z)(b) = F(z)(\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}) = z(p_{\partial_0 b})^{-1} z(b) z(p_{\partial_1 b}),$$

and given a 0-simplex a we have

$$(C \cdot F)(f) = C(f_{a_0}) = c(f_{a_0})$$

Define $u(z)_a \equiv z(p_a)$ for $a \in \Sigma_0(K)$. It can be easily seen that the mapping $Z^1(K, G) \ni z \rightarrow u(z)$ defines a natural isomorphism between the functor $C \cdot F$ and the functor $\text{id}_{Z^1(K, G)}$. \square

Observe in particular that the group homomorphism corresponding to the trivial 1-cocycle ι is the trivial one, namely $\sigma([p]) = e$ for any $[p] \in \pi_1(K, a_0)$. Hence, a 1-cocycle of $Z^1(K, G)$ is a 1-coboundary if, and only

if, the corresponding group homomorphism $F(z)$ is equivalent to the trivial one. In particular if K is simply connected, then $Z^1(K, G) = B^1(K, G)$ for any group G .

The existence of 1-cocycles, which are not 1-coboundaries, relies, in particular, on the following corollary

Corollary 3.9. *Let M be a nonempty, Hausdorff and arcwise connected topological space which admits a base for the topology consisting of arcwise and simply connected subsets of M . Let K denote the poset formed by such a base ordered under inclusion \subseteq . Then*

$$H(\pi_1(M, x_0), G) \cong H(\pi_1(K, a_0), G) \cong Z^1(K, G) ,$$

for any $x_0 \in M$ and $a_0 \in \Sigma_0(K)$ with $x_0 \in a_0$, where the symbol \cong means equivalence of categories.

Proof. $\pi_1(M, x_0)$ is isomorphic to $\pi_1(K, a_0)$ (see in Section 2). As observed at the beginning of this section, this entails that $H(\pi_1(M, x_0), G)$ and $H(\pi_1(K, a_0), G)$ are isomorphic categories. Therefore the proof follows by Proposition 3.8. \square

Let M be a nonsimply connected topological space and let K be a basis for the topology of M as defined in the statement of Corollary 3.9. Then to any nontrivial group homomorphism in $H(\pi_1(M, x_0), G)$ there corresponds a 1-cocycle of $Z^1(K, G)$ which is not a 1-coboundary.

4 Connections

This section is entirely devoted to studying connections and related notions like the curvature, holonomy group and the central connections. We will show how connections and 1-cocycles are related, thus allowing one to interpret a 1-cocycle as a principal bundle and a connection 1-cochain as the connection of this principal bundle. We will prove the existence of nonflat connections, a “poset” version of the Ambrose-Singer Theorem, and that to any flat connection with values in G , there corresponds a group homomorphism from the fundamental group of the poset into G .

4.1 Connections and curvature

We now give the definition of a connection of a poset with values in a group. To this end, recall the definition of the set $\Sigma_n^{\text{inf}}(K)$ of inflating n -simplices (see Section 2).

Definition 4.1. A 1-cochain u of $C^1(K, G)$ is said to be a **connection** if it satisfies the following properties:

- (i) $u(\bar{b}) = u(b)^{-1}$ for any $b \in \Sigma_1(K)$;
- (ii) $u(\partial_0 c) u(\partial_2 c) = u(\partial_1 c)$, for any $c \in \Sigma_2^{\text{inf}}(K)$.

We denote the set of connection 1-cochains with values in G by $U^1(K, G)$.

This definition of a connection is related to the notion of the link operator in a lattice gauge theory ([3]) and to the notion of a generalized connection in loop quantum gravity ([1, 7]). Both the link operator and the generalized connection can be seen as a mapping A which associates an element $A(e)$ of a group G to any oriented edge e of a graph α , and enjoying the following properties

$$A(\bar{e}) = A(e)^{-1}, \quad A(e_2 * e_1) = A(e_2) A(e_1), \quad (43)$$

where, \bar{e} is the reverse of the edge e ; $e_2 * e_1$ is the composition of the edges e_1, e_2 obtained by composing the end of e_1 with the beginning of e_2 . Now, observe that to any poset K there corresponds an oriented graph $\alpha(K)$ whose set of vertices is $\Sigma_0(K)$, and whose set of edges is $\Sigma_1(K)$. Then, by property (i) of the above definition and property (32), any connection $u \in U^1(K, G)$ defines a mapping from the edges of $\alpha(K)$ to G satisfying (43). The new feature of our definition of connection, is to require property (ii) in Definition 4.1, thus involving the poset structure. The motivation for this property will become clear in the next section: thanks to this property any connection u can be seen as a connection on the principal bundle described by a 1-cocycle (see Theorem 4.8).

Let us now observe that any 1-cocycle is a connection. Furthermore, if u is a connection then so is any equivalent 1-cochain (the proof is similar to the proof of the same property for 1-cocycles, see Section 3.2).

Lemma 4.2. Let $\gamma : G_1 \rightarrow G$ be a group homomorphism. Given $v \in C^1(K, G_1)$ consider $\gamma \circ v \in C^1(K, G)$. Then: if v is a connection then $\gamma \circ v$ is a connection; the converse holds if γ is injective.

Proof. Clearly, if v is a connection so is $\gamma \circ v$. Conversely, assume that γ is injective and that $\gamma \circ v$ is a connection. If $c \in \Sigma_2^{\text{inf}}(K)$, then

$$\gamma(v(\partial_0 c) v(\partial_2 c)) = \gamma \circ v(\partial_0 c) \gamma \circ v(\partial_2 c) = \gamma \circ v(\partial_1 c) = \gamma(v(\partial_1 c)),$$

hence $v(\partial_0 c) v(\partial_2 c) = v(\partial_1 c)$, since γ is injective. Furthermore, for any 1-simplex b we have

$$\gamma(v(\bar{b})) = \gamma \circ v(\bar{b}) = (\gamma \circ v(b))^{-1} = \gamma(v(b)^{-1}).$$

So, as γ is injective, we have $v(\bar{b}) = v(b)^{-1}$, and this entails that v is a connection. \square

Lemma 4.3. *Given $u \in U^1(K, G)$, then*

- (a) $u(\sigma_0(a_0)) = e$ for any $a_0 \in \Sigma_0(K)$,
- (b) $u(b) = u(b_1)$ for any $b, b_1 \in \Sigma_1^{\text{inf}}(K)$ with $\partial b = \partial b_1$.

Proof. (a) Since a degenerate 1-simplex is an inflating 1-simplex, by Definition 4.1(ii) we have

$$u(\sigma_0(a_0)) u(\sigma_0(a_0)) = u(\sigma_0(a_0)) \iff u(\sigma_0(a_0)) = e.$$

(b) Given b_1 and b as in the statement. Since b, b_1 are inflating 1-simplices, the 1-simplex b_0 defined as $\partial b_0 \equiv \partial b$, and $|b_0| \equiv \partial_0 b$, is inflating. Moreover $\partial b_0 = \partial b = \partial b_1$ and $|b_0| \subseteq |b|, |b_1|$. As $|b_0| \subseteq |b|$, the 2-simplex c defined by

$$\partial_0 c \equiv \sigma_0(\partial_0 b), \quad \partial_1 c \equiv b, \quad \partial_2 c \equiv b_0, \quad |c| \equiv |b|$$

is an inflating 2-simplex. By Definition 4.1(ii) and by (a) we have

$$u(b_0) = u(\sigma_0(\partial_0 b)) u(b_0) = u(\partial_0 c) u(\partial_2 c) = u(\partial_1 c) = u(b).$$

The same reasoning leads to $u(b_0) = u(b_1)$, hence $u(b) = u(b_1)$. \square

In words, this lemma says that connections act trivially on degenerate 1-simplices, and that their values do not depend on the support of the inflating 1-simplices.

We call the full subcategory of $C^1(K, G)$ whose set of objects is $U^1(K, G)$ *the category of connection 1-cochains* with values in G . It will be denoted by the same symbol $U^1(K, G)$ as used to denote the corresponding set of objects. Note that $Z^1(K, G)$ is a full subcategory of $U^1(K, G)$. Furthermore, if $\gamma : G_1 \rightarrow G$ is a group homomorphism, by Lemma 4.2, the restriction of the functor $\gamma \circ$ to $U^1(K, G_1)$ defines a functor from $U^1(K, G_1)$ into $U^1(K, G)$.

As observed, any 1-cocycle is a connection. The converse does not hold, in general, and the obstruction is a 2-coboundary.

Definition 4.4. The **curvature** of a connection $u \in U^1(K, G)$ is the 2-coboundary $W_u \equiv du \in B^2(K, G)$. Explicitly, by using relation (25) we have

$$\begin{aligned} (W_u)_1(b) &= (e, \text{ad}(u(b))), & b \in \Sigma_1(K), \\ (W_u)_2(c) &= (w_u(c), \text{ad}(u(\partial_1 c))) & c \in \Sigma_2(K), \end{aligned}$$

where $w_u : \Sigma_2(K) \rightarrow G$ defined as

$$w_u(c) \equiv u(\partial_0 c) u(\partial_2 c) u(\partial_1 c)^{-1}, \quad c \in \Sigma_2(K).$$

A connection $u \in U^1(K, G)$ is said to be **flat** whenever its curvature is trivial i.e. $W_u \in (2G)_1$ or, equivalently, if $w_u(c) = e$ for any 2-simplex c .

We now draw some consequences of our definition of the curvature of a connection and point out the relations of this notion to the corresponding one in the theory of principal bundles.

First, note that a connection u is flat if, and only if, u is a 1-cocycle. Then, as an immediate consequence of Proposition 3.8, we have a poset version of a classical result of the theory of principal bundles [5, 4].

Corollary 4.5. *There is, up to equivalence, a 1-1 correspondence between flat connections of K with values in G and group homomorphisms from $\pi_1(K)$ into G .*

The existence of nonflat connections will be shown in Section 4.4 where examples will be given.

Secondly, in a principal bundle the curvature form is the covariant exterior derivative of a connection form, namely the 2-form with values in the Lie algebra of the group, obtained by taking the exterior derivative of the connection form and evaluating this on the horizontal components of pairs of vectors of the tangent space (see [5]). Although, no differential structure is present in our approach, but W_u encodes this type of information. In fact, given a connection u , if we interpret $u(p)$ as the horizontal lift of a path p , then the equation

$$w_u(c) u(\partial_1 c) = u(\partial_0 c * \partial_2 c) w_u(c), \quad c \in \Sigma_2(K), \quad (44)$$

may be understood as saying that $w_u(c)$ intertwines the horizontal lift of the path $\partial_1 c$ and that of the path $\partial_0 c * \partial_2 c$.

Thirdly, the structure equation of the curvature form (see [5]) says that the curvature equals the exterior derivative of the connection form plus the

commutator of the connection form. Notice that the second component $(W_u)_2$ of the curvature can be rewritten as

$$(W_u)_2(c) = (w_u(c), \text{ad}(w_u(c))^{-1}) \times (e, \text{ad}(u(\partial_0 c)u(\partial_2 c))), \quad (45)$$

for any 2-simplex c , where \times is the composition (14) of the 2-category $2G$. This, equation represents, in our formalism, the structure equation of the curvature with $(w_u(c), \text{ad}(w_u(c))^{-1})$ in place of the exterior derivative, and $(\iota, \text{ad}(u(\partial_0 c)u(\partial_2 c)))$ in place of the commutator of the connection form.

Fourthly, as a consequence of Lemma 3.1 we have that W_u is a 2-cocycle. The 2-cocycle identity that is $dW_u \in (3G)_2$ or, equivalently,

$$w_u(\partial_0 d) w_u(\partial_2 d) = \text{ad}(u(\partial_0 d)) (w_u(\partial_3 d)) w_u(\partial_1 d), \quad \Sigma_3(K), \quad (46)$$

corresponds, in our framework, to the Bianchi identity.

We conclude with the following result.

Lemma 4.6. *For any connection u the following assertions hold:*

- (a) $w_u(c^{(01)}) = w_u(c)^{-1}$ for any 2-simplex c ;
- (b) $w_u(c) = e$ if c is either a degenerate or an inflating 2-simplex.

Proof. (a) follows directly from the definition of $c^{(01)}$, see Section 2. (b) If c is an inflating 2-simplex, then $w_u(c) = e$ because of Definition 4.1(ii). Given a 1-simplex b , then

$$\begin{aligned} w_u(\sigma_0(b)) &= u(\partial_0 \sigma_0(b)) u(\partial_2 \sigma_0(b)) u(\partial_1 \sigma_0(b))^{-1} \\ &= u(b) u(\sigma_0(\partial_1 b)) u(b)^{-1} = e, \end{aligned}$$

because by Lemma 4.3(a) we have that $u(\sigma_0(\partial_1 b)) = e$. Analogously we have that $w_u(\sigma_1(b)) = e$. \square

In words, statement (b) asserts that the curvature of a connection is trivial when restricted to inflating simplices.

Remark 4.7. It is worth pointing out some analogies between the theory of connections, as presented in this paper, and that developed in synthetic geometry by A. Kock [6], and in algebraic topology by L. Breen and W. Messing [2]. The contact point with our approach resides in the fact that both of the other approaches make use of a combinatorial notion of differential forms taking values in a group G . So in both cases connections turn out to be combinatorial 1-forms. Concerning the curvature, the definition of W_u is formally the same as the definition of curving data given in [2], since

this is the 2-coboundary of a connection, taking values in a 2-category associated with G . Whereas, in [6] the curvature is the 2-coboundary of a connection, taking values in G , and is formally the same as w_u . The only difference to these other two approaches is that in our case w_u is not invariant under oriented equivalence of 2-simplices (examples of connections having this feature will be given at the end of Section 4.4).

4.2 The cocycle induced by a connection

We analyze the relation between connections and 1-cocycles more deeply. The main result is that to any connection there corresponds a unique 1-cocycle. This, on the one hand, confirms the interpretation of 1-cocycles as principal bundles. On the other hand this result will allow us to construct examples of nonflat connections in the next section.

Theorem 4.8. *For any $u \in U^1(K, G)$, there exists a unique 1-cocycle $z \in Z^1(K, G)$ such that $u(b) = z(b)$ for any 1-simplex $b \in \Sigma_1^{\text{inf}}(K)$.*

Proof. Within this proof we adopt the following notation: for any $\mathcal{O} \in U_a \cap U_{a_1}$ the 3-tuple $(\mathcal{O}; a, a_1)$ denotes the 1-simplex with support \mathcal{O} , 0-boundary a and 1-boundary a_1 . Consider the open set U_a (12) of the fundamental covering of K , and define

$$z_{a_1, a}(\mathcal{O}) \equiv u(\mathcal{O}; \mathcal{O}, a_1)^{-1} u(\mathcal{O}; \mathcal{O}, a), \quad \mathcal{O} \in U_a \cap U_{a_1}. \quad (47)$$

So, we have a family of functions $z_{a_1, a} : U_a \cap U_{a_1} \rightarrow G$. Let $\mathcal{O}_1 \subseteq \mathcal{O}$, with $\mathcal{O}, \mathcal{O}_1 \in U_a \cap U_{a_1}$. By using the defining properties of connection we have

$$\begin{aligned} z_{a_1, a}(\mathcal{O}) &= u(\mathcal{O}; \mathcal{O}, a_1)^{-1} u(\mathcal{O}; \mathcal{O}, a) \\ &= (u(\mathcal{O}; \mathcal{O}, \mathcal{O}_1) u(\mathcal{O}_1; \mathcal{O}_1, a_1))^{-1} u(\mathcal{O}; \mathcal{O}, \mathcal{O}_1) u(\mathcal{O}_1; \mathcal{O}_1, a) \\ &= u(\mathcal{O}_1; \mathcal{O}_1, a_1)^{-1} u(\mathcal{O}_1; \mathcal{O}_1, a) \\ &= z_{a_1, a}(\mathcal{O}_1). \end{aligned}$$

If $a \subseteq a_1$ and $\mathcal{O} \in U_{a_1}$, then

$$\begin{aligned} z_{a_1, a}(\mathcal{O}) &= u(\mathcal{O}; \mathcal{O}, a_1)^{-1} u(\mathcal{O}; \mathcal{O}, a) = u(\mathcal{O}; \mathcal{O}, a_1)^{-1} u(\mathcal{O}; \mathcal{O}, a_1) u(\mathcal{O}; a_1, a) \\ &= u(\mathcal{O}; a_1, a) \end{aligned}$$

Assume that $\mathcal{O} \in U_a \cap U_{a_1} \cap U_{a_2}$. Then

$$\begin{aligned} z_{a_2, a_1}(\mathcal{O}) z_{a_1, a}(\mathcal{O}) &= u(\mathcal{O}; \mathcal{O}, a_2)^{-1} u(\mathcal{O}; \mathcal{O}, a_1) u(\mathcal{O}; \mathcal{O}, a_1)^{-1} u(\mathcal{O}; \mathcal{O}, a) \\ &= u(\mathcal{O}; \mathcal{O}, a_2)^{-1} u(\mathcal{O}; \mathcal{O}, a) = z_{a_2, a}(\mathcal{O}). \end{aligned}$$

Summing up, to a connection u corresponds a family of functions $z_{a_1,a} : U_a \cap U_{a_1} \rightarrow G$ satisfying the following properties:

$$\begin{aligned} (i) \quad & z_{a_1,a}(\mathcal{O}) = z_{a_1,a}(\mathcal{O}_1), & \mathcal{O}_1 \subseteq \mathcal{O}, \quad \mathcal{O}, \mathcal{O}_1 \in U_a \cap U_{a_1}; \\ (ii) \quad & z_{a_1,a}(\mathcal{O}) = u(\mathcal{O}; a_1, a), & \mathcal{O} \in U_{a_1}, \quad a \subseteq a_1; \\ (iii) \quad & z_{a_2,a_1}(\mathcal{O}) z_{a_1,a}(\mathcal{O}) = z_{a_2,a}(\mathcal{O}), & \mathcal{O} \in U_a \cap U_{a_1} \cap U_{a_2}. \end{aligned} \quad (48)$$

Now, note that since $\partial_0 b, \partial_1 b \leq |b|$, we have that $|b| \in U_{\partial_0 b} \cap U_{\partial_1 b}$. Hence we can define

$$z(b) \equiv z_{\partial_0 b, \partial_1 b}(|b|), \quad b \in \Sigma_1(K). \quad (49)$$

Given a 2-simplex c . By using properties (i)–(iii) we have

$$\begin{aligned} z(\partial_0 c) z(\partial_2 c) &= z_{\partial_{00} c, \partial_{10} c}(|\partial_0 c|) z_{\partial_{02} c, \partial_{12} c}(|\partial_2 c|) \\ &= z_{\partial_{00} c, \partial_{10} c}(|c|) z_{\partial_{02} c, \partial_{12} c}(|c|) \\ &= z_{\partial_{01} c, \partial_{10} c}(|c|) z_{\partial_{10} c, \partial_{11} c}(|c|) \\ &= z_{\partial_{01} c, \partial_{11} c}(|c|) \\ &= z(\partial_1 c). \end{aligned}$$

Hence z is 1-cocycle. Moreover, if b is an inflating 1-simplex, then

$$\begin{aligned} z(b) &= z_{\partial_0 b, \partial_1 b}(|b|) = u(|b|; |b|, \partial_0 b)^{-1} u(|b|; |b|, \partial_1 b) \\ &= u(|b|; |b|, \partial_0 b)^{-1} u(|b|; |b|, \partial_0 b) u(|b|; \partial_0 b, \partial_1 b) = u(|b|; \partial_0 b, \partial_1 b) \\ &= u(b). \end{aligned}$$

z is clearly the unique 1-cocycle with $z(b) = u(b)$ for $b \in \Sigma_1^{\text{inf}}(K)$. \square

On the basis of this theorem, we can introduce the following definition.

Definition 4.9. A connection $u \in U^1(K, G)$ is said to **induce** the 1-cocycle $z \in Z^1(K, G)$ whenever

$$u(b) = z(b), \quad b \in \Sigma_1^{\text{inf}}(K).$$

We denote the set of connections of $U^1(K, G)$ inducing the 1-cocycle z by $U^1(K, z)$.

The geometrical meaning of $U^1(K, z)$ is the following: just as a 1-cocycle z stands for a principal bundle over K so the set of connections $U^1(K, z)$ stands for the set of connections on that principal bundle. Theorem 4.8 says that the set of connections with values in G is partitioned as

$$U^1(K, G) = \dot{\cup} \{U^1(K, z) \mid z \in Z^1(K, G)\} \quad (50)$$

where the symbol $\dot{\cup}$ means disjoint union.

Lemma 4.10. *Given $z_1, z \in Z^1(K, G)$, let $u_1 \in U^1(K, z_1)$ and $u \in U^1(K, z)$. Then $(u_1, u) \subseteq (z_1, z)$. In particular if $u_1 \cong u$, then $z_1 \cong z$.*

Proof. (a) By equations (47) and (49), we have

$$z(\mathcal{O}; a_1, a) = u(\mathcal{O}; \mathcal{O}, a_1)^{-1} u(\mathcal{O}; \mathcal{O}, a),$$

for any 1-simplex $(\mathcal{O}; a_1, a)$. The same holds for z_1 and u_1 . Given $f \in (u_1, u)$, we have

$$\begin{aligned} f_{a_1} z_1(\mathcal{O}; a_1, a) &= f_{a_1} u_1(\mathcal{O}; a_1, \mathcal{O}) u_1(\mathcal{O}; \mathcal{O}, a) \\ &= u(\mathcal{O}; a_1, \mathcal{O}) f_{\mathcal{O}} u_1(\mathcal{O}; \mathcal{O}, a) \\ &= u(\mathcal{O}; \mathcal{O}, a_1)^{-1} u(\mathcal{O}; \mathcal{O}, a) f_a = z(\mathcal{O}; a_1, a) f_a, \end{aligned}$$

where we have use the fact that $(\mathcal{O}; a_1, \mathcal{O})$ is the reverse of $(\mathcal{O}; \mathcal{O}, a_1)$. Hence $f \in (z_1, z)$, and this completes the proof. \square

Now, given a 1-cocycle $z \in Z^1(K, G)$, we call the category of *connections inducing z* , the full subcategory of $U^1(K, G)$ whose objects belong to $U^1(K, z)$. As it is customary in this paper, we denote this category by the same symbol $U^1(K, z)$ as used to denote the corresponding set of objects.

Lemma 4.11. *Let $z \in Z^1(K, G_1)$ and let $\gamma : G_1 \rightarrow G$ be an injective group homomorphism. Then, the functor $\gamma \circ : U^1(K, z) \rightarrow U^1(K, \gamma \circ z)$ is injective and faithful.*

Proof. Given $u \in U^1(K, z)$, it is easy to see that $\gamma \circ u \in U^1(K, \gamma \circ z)$. Clearly, as γ is injective, the functor $\gamma \circ$ is injective and faithful. \square

We note the following simple result.

Lemma 4.12. *If $z_1 \cong z$, then the categories $U^1(K, z_1)$ and $U^1(K, z)$ are equivalent.*

Assume that K is simply connected. In this case any 1-cocycle is a 1-coboundary (see Section 3.3). Then the category $U^1(K, z)$ is equivalent to $U^1(K, \iota)$ for any $z \in Z^1(K, G)$.

4.3 Central connections

We now briefly study the family of central connections, whose main feature, as we will show below, is that any such connection can be uniquely decomposed as the product of the induced cocycle by a suitable connection taking values in the centre of the group.

Definition 4.13. A connection $u \in U^1(K, G)$ is said to be a **central connection** whenever the component w_u of the curvature W_u takes values in the centre $\mathcal{Z}(G)$. We denote the set of central connections by $U_{\mathcal{Z}}^1(K, G)$.

Let us start to analyze the properties of central connections. Clearly 1-cocycles are central connections. However, the main property that can be directly deduced from the above definition is that the component w_u of the curvature W_u of a central connection u is invariant under oriented equivalence of 2-simplices. In fact by the definition of w_u , it is easily seen that

$$w_u(c) = w_u(c_1) = w_u(c_2)^{-1}, \quad c_1 \in [c], c_2 \in \overline{[c]}, \quad (51)$$

for any 2-simplex c , where $[c]$ and $\overline{[c]}$ are, respectively, the classes of 2-simplices having the same and the reversed orientation of c .

Proposition 4.14. A connection u of $U^1(K, G)$ is central if, and only if, it can be uniquely decomposed as

$$u(b) = z_u(b) \chi_u(b), \quad b \in \Sigma_1(K),$$

where $z_u \in Z^1(K, G)$, and $\chi_u \in U^1(K, \iota)$ with values in $\mathcal{Z}(G)$.

Proof. (\Leftarrow) Assume that a connection u admits a decomposition as in the statement. Since χ_u takes values in the centre, so does w_u . Furthermore, since $\chi_u \in U^1(K, \iota)$ then $\chi_u(b) = e$ for any inflating 1-simplex b . This entails that z_u is nothing but the 1-cocycle induced by u . This is enough for uniqueness. (\Rightarrow) Assume that u is central. For any 1-simplex b let c_b denote the 2-simplex defined as

$$\partial_1 c_b \equiv b, \quad \partial_0 c_b \equiv (|b|, \partial_1 b, |b|), \quad \partial_2 c_b \equiv (|b|, |b|, \partial_0 b), \quad |c_b| \equiv |b|.$$

As w_u takes values in $\mathcal{Z}(G)$ we have

$$u(b) w_u(c_b) = u(\partial_1 c_b) w_u(c_b) = u(\partial_0 c_b) u(\partial_2 c_b) = z_u(b),$$

where the latter identity is a consequence of the fact that $u(\partial_0 c_b) u(\partial_2 c_b)$ is nothing but the definition (47) of the 1-cocycle induced by u . Now, define

$$\chi_u(b) \equiv w_u(c_b), \quad b \in \Sigma_1(K).$$

Since $\chi_u(b) = u(b) z_u(b)^{-1}$, one can easily deduce that $\chi_u \in U^1(K, \iota)$, and this completes the proof. \square

As a consequence of this result the set $U_{\mathcal{Z}}^1(K, z)$ of central connections inducing the 1-cocycle z , has the structure of an Abelian group. In fact, given $u, u_1 \in U_{\mathcal{Z}}^1(K, z)$, define

$$u \star_z u_1(b) \equiv u(b) z(b)^{-1} u_1(b), \quad b \in \Sigma_1(K). \quad (52)$$

By Proposition 4.14, we have $u \star_z u_1(b) = z(b) \chi_u(b) \chi_{u_1}(b)$ for any 1-simplex b . This entails that

$$u \star_z u_1 = u_1 \star_z u \quad \text{and} \quad u \star_z u_1 \in U_{\mathcal{Z}}^1(K, z).$$

Finally, it can be easily seen that $U_{\mathcal{Z}}^1(K, z)$ with \star_z is an Abelian group whose identity is z , and such that the inverse of a connection u is the connection defined as $z(b) \chi_u(b)^{-1}$ for any 1-simplex b .

Finally, in Section 4.1 we pointed out the analogy between equation (45) and the structure equation of the curvature of a connection in a principal bundle. This analogy is stronger for a central connection u since we have

$$\begin{aligned} (W_u)_2(c) &= (w_u(c), \iota) \times (e, \text{ad}(u(\partial_0 c)u(\partial_2 c))) \\ &= (e, \text{ad}(u(\partial_0 c)u(\partial_2 c))) \times (w_u(c), \iota), \end{aligned} \quad (53)$$

for any 2-simplex c . Hence, as for principal bundles, equation (45) for a central connection is symmetric with respect to the interchange of the two factors.

4.4 Existence of nonflat connections

We investigate the existence of nonflat connections. As a first step, we show that there is a very particular class of posets which not admitting nonflat connections.

Recall that a poset K is said to be *totally ordered* whenever for any pair $\mathcal{O}, \mathcal{O}_1 \in K$ either $\mathcal{O} \leq \mathcal{O}_1$ or $\mathcal{O}_1 \leq \mathcal{O}$. Clearly, a totally ordered poset is directed and, consequently, pathwise connected (it is also simply connected, see Section 2).

Corollary 4.15. *If K is totally ordered, any connection is flat.*

Proof. If K is totally ordered and b is any 1-simplex either b or \bar{b} is an inflating 1-simplex. Hence, by Theorem 4.8 any connection coincides with the associated 1-cocycle. \square

Another obvious situation where nonflat connections do not exist is when the group of coefficients G is trivial, i.e. $G = e$. Two observations on these results are in order. First, Corollary 4.15 cannot be directly deduced from the definition of a connection. Secondly, as explained earlier, these two situations never arise in the applications we have in mind.

Now, our purpose is to show that except when the poset is totally directed or the group of coefficients is trivial, nonflat connections always exist. Let us starting by the following

Lemma 4.16. *Assume that there exists a 1-cochain $v \in C^1(K, G)$ such that $v(b) = e = v(\bar{b})$ for any inflating 1-simplex b . Then, for any 1-cocycle $z \in Z^1(K, G)$ the 1-cochain $v(z)$ defined as*

$$v(z)(b) \equiv v(\bar{b})^{-1} z(b) v(b), \quad b \in \Sigma_1(K), \quad (54)$$

is a connection inducing z .

Proof. By the definition of v for any inflating 1-simplex b we have that

$$v(z)(b) = v(\bar{b})^{-1} z(b) v(b) = e z(b) e = z(b) .$$

This, in particular, entails that $v(z)$ satisfies property (ii) of the definition of connections. For any 1-simplex b we have

$$\begin{aligned} v(z)(\bar{b}) &= v(\bar{\bar{b}})^{-1} z(\bar{b}) v(\bar{b}) = v(b)^{-1} z(b)^{-1} v(\bar{b}) \\ &= (v(\bar{b})^{-1} z(b) v(b))^{-1} = v(z)(b)^{-1}. \end{aligned}$$

Hence $v(z) \in U^1(K, z)$. □

It is very easy to prove the existence of elements of $C^1(K, G)$ satisfying the properties of the statement. For instance, given a 1-simplex b , define

$$v(b) \equiv \begin{cases} e & b \text{ or } \bar{b} \in \Sigma_1^{\text{inf}}(K) \\ g(b) & \text{otherwise,} \end{cases} \quad (55)$$

where $g(b)$ is some element of the group G . So v is a 1-cochain satisfying the relation $v(b) = e = v(\bar{b})$ for any inflating 1-simplex b .

Now, assume that K is a pathwise connected but not totally directed poset. Let G be a nontrivial group. Let $v \in C^1(K, G)$ be defined by (55), and let $z \in Z^1(K, G)$. Consider the connection $v(z) \in U^1(K, z)$. We want to find conditions on v implying that $v(z)$ is not flat.

As $v(z) \in U^1(K, z)$, Theorem 4.8 says that if $v(z)$ is flat then $v(z) = z$.

Hence, $v(z)$ is not flat if, and only if, it differs from z on a 1-simplex b such that both b and \bar{b} are noninflating. Then

$$\begin{aligned} v(z)(b) \neq z(b) &\iff v(\bar{b})^{-1} z(b) v(b) \neq z(b) \\ &\iff z(b) v(b) \neq v(\bar{b}) z(b) \\ &\iff z(b) g(b) \neq g(\bar{b}) z(b) \end{aligned}$$

So, for instance, if we take

$$g(b) = z(b)^{-1}, \quad g(\bar{b}) = g z(b)^{-1} \text{ with } g \neq e,$$

then $v(z)$ is not flat. Note that the above choice is always possible because G is nontrivial by assumption. In conclusion we have shown the following

Theorem 4.17. *Let K be a pathwise connected but not totally directed poset. Let G be a nontrivial group. Then for any 1-cocycle $z \in Z^1(K, G)$ there are connections in $U^1(K, z)$ which are not flat.*

Concerning central connections, in the case that K and G satisfy the hypotheses of the statement of Theorem 4.17, and the centre of the group G is nontrivial, then by using the above reasoning it is very easy to prove the existence of nonflat central connections.

4.5 Holonomy and reduction of connections

Consider a connection u of $U^1(K, G)$. Fix a base 0-simplex a_0 and define

$$H_u(a_0) \equiv \{u(p) \in G \mid p \in K(a_0)\} , \quad (56)$$

recalling that $K(a_0)$ is the set of loops of K with endpoint a_0 . By the defining properties of connection 1-cochains it is very easy to see that $H_u(a_0)$ is a subgroup of G . Furthermore let

$$H_u^0(a_0) \equiv \{u(p) \in G \mid p \in K(a_0), p \sim \sigma_0(a_0)\} , \quad (57)$$

where $p \sim \sigma_0(a_0)$ means that p is homotopic to the degenerate 1-simplex $\sigma_0(a_0)$. In this case, too, it is easy to see that $H_u^0(a_0)$ is a subgroup of G . Moreover, since $p * q * \bar{p} \sim \sigma_0(a_0)$ whenever $q, p \in K(a_0)$ and $q \sim \sigma_0(a_0)$, $H_u^0(a_0)$ is a normal subgroup of $H_u(a_0)$. $H_u(a_0)$ and $H_u^0(a_0)$ are called respectively *the holonomy and the restricted holonomy group of u based on a_0* .

As K is pathwise connected, we have the following

Lemma 4.18. *Given $u \in U^1(K, G)$, let $\gamma : G_1 \rightarrow G$ be an injective homomorphism. The following assertions hold.*

- (a) $H_u(a_0)$ and $H_u(a_1)$ are conjugate subgroups of G for any $a_0, a_1 \in \Sigma_0(K)$.
- (b) Given $u_1 \in U^1(K, G_1)$. If $\gamma \circ u_1$ is equivalent to u , then the holonomy groups $H_{u_1}(a_0)$ and $H_u(a_0)$ are isomorphic.

The same assertions hold for the restricted holonomy groups.

Proof. (a) Let p be a path from a_0 to a_1 . For any $g \in H_u(a_0)$, there is a loop $q \in K(a_0)$ such that $g = u(q)$. Observe that $p * q * \bar{p} \in K(a_1)$, hence $u(p) g u(p)^{-1} = u(p * q * \bar{p}) \in H_u(a_1)$. By the symmetry of the reasoning, $H_u(a_0) \ni g \rightarrow u(p) g u(p)^{-1} \in H_u(a_1)$ is a group isomorphism. (b) Let $u_1 \in U^1(K, G_1)$ and let $f \in (\gamma \circ u_1, u)$. Since for any loop $p \in K(a_0)$, $f_{a_0} \gamma \circ u_1(p) = u(p) f_{a_0}$, the map $H_{u_1}(a_0) \ni g \rightarrow f_{a_0} \gamma(g) f_{a_0}^{-1} \in H_u(a_0)$ is a group isomorphism. \square

We now prove an analogue of the Ambrose-Singer theorem for connections of a poset.

Theorem 4.19. *Let $u \in U^1(K, G)$, $a_0 \in \Sigma_0(K)$ and let ι be the inclusion of $H_u(a_0)$ in G . Then there exists $u_1 \in U^1(K, H_u(a_0))$ such that $\iota \circ u_1 \cong u$.*

Proof. For any 0-simplex a , let p_a be a path from a_0 to a . Then define

$$u_1(b) \equiv u(\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}), \quad b \in \Sigma_1(K). \quad (58)$$

Note that $u_1(b) \in H_u(a_0)$ for any 1-simplex b because $\overline{p_{\partial_0 b}} * b * p_{\partial_1 b} \in K(a_0)$. Secondly, for any 1-simplex b we have

$$u_1(\bar{b}) = u(\overline{p_{\partial_1 b}} * \bar{b} * p_{\partial_0 b}) = u(\overline{\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}}) = u_1(b)^{-1}.$$

Thirdly, let $c \in \Sigma_2^{\text{inf}}(K)$. Then

$$\begin{aligned} u_1(\partial_0 c) u_1(\partial_2 c) &= u(\overline{p_{\partial_{00} c}} * \partial_0 c * p_{\partial_{10} c}) u(\overline{p_{\partial_{02} c}} * \partial_2 c * p_{\partial_{12} c}) \\ &= u(\overline{p_{\partial_{00} c}}) u(\partial_0 c) u(p_{\partial_{10} c}) u(\overline{p_{\partial_{02} c}}) u(\partial_2 c) u(p_{\partial_{12} c}) \\ &= u(\overline{p_{\partial_{01} c}}) u(\partial_0 c) u(\partial_2 c) u(p_{\partial_{11} c}) \\ &= u(\overline{p_{\partial_{01} c}}) u(\partial_1 c) u(p_{\partial_{11} c}) \\ &= u_1(\partial_1 c). \end{aligned}$$

Therefore we have that $u_1 \in U^1(K, H_u(a_0))$. Finally, for any 0-simplex a let $f_a \equiv u(p_a)$. Then for any 1-simplex b we have

$$\begin{aligned} f_{\partial_0 b} u_1(b) &= u(p_{\partial_0 b}) u(\overline{p_{\partial_0 b}} * b * p_{\partial_1 b}) \\ &= u(p_{\partial_0 b}) u(\overline{p_{\partial_0 b}}) u(b) u(p_{\partial_1 b}) = u(b) u(p_{\partial_1 b}) \\ &= u(b) f_{\partial_1 b}, \end{aligned}$$

namely $f \in (\iota \circ u_1, u)$. Thus $\iota \circ u_1 \simeq u$. \square

5 Gauge transformations

In the previous sections we have given several results to support the interpretation of 1-cocycles of a poset as principal bundles over the poset. As the final issue of the present paper, we now introduce what we mean by the group of gauge transformations of a 1-cocycle.

Given a 1-cocycle z of $Z^1(K, G)$, define

$$\mathcal{G}(z) \equiv (z, z). \quad (59)$$

An element of $\mathcal{G}(z)$ will be denoted by g . The composition law between morphisms of 1-cochains endows $\mathcal{G}(z)$ with a structure of a group. The identity e of this group is given by $e_a = e$ for any 0-simplex a . The inverse g^{-1} of an element $g \in \mathcal{G}(z)$ is obtained by composing g with the inverse of G . We call $\mathcal{G}(z)$ the *group of gauge transformations of z* .

Lemma 5.1. *If $z \in B^1(K, G)$, then $\mathcal{G}(z) \cong G$.*

Proof. Observe that, since K is connected, $\mathcal{G}(z)$ is the set of constant functions from $\Sigma_0(K)$ to G and hence is isomorphic to G . As z is a 1-coboundary, it is equivalent to the trivial 1-cocycle ι , i.e. there exists an $f \in (z, \iota)$. The mapping $\mathcal{G}(z) \ni g \mapsto f^{-1} g f \in \mathcal{G}(\iota)$ is a group isomorphism. \square

As a consequence of this lemma and Proposition 3.8, if the poset is simply connected then $\mathcal{G}(z) \cong G$ for any 1-cocycle z . This is also the case when G is Abelian.

Lemma 5.2. *If G is Abelian, then $\mathcal{G}(z) \cong G$ for any $z \in Z^1(K, G)$.*

Proof. For any $g \in \mathcal{G}(z)$ and for any 1-simplex b we have

$$g_{\partial_1 b} z(b) = z(b) g_{\partial_0 b} = g_{\partial_0 b} z(b) .$$

Hence $g_{\partial_1 b} = g_{\partial_0 b}$ for any 1-simplex b . Since K is pathwise connected, $g_a = g$ for any 0-simplex a . \square

Thus, for Abelian groups, the action of the group of gauge transformations is always *global*, that is independent of the 0-simplex.

Given a 1-cocycle $z \in Z^1(K, G)$ consider the group $\mathcal{G}(z)$ of gauge transformations of z . For any $u \in U^1(K, z)$ and $g \in \mathcal{G}(z)$, define

$$\alpha_g(u)(b) \equiv g_{\partial_0 b} u(b) g_{\partial_1 b}^{-1}, \quad b \in \Sigma_1(K). \quad (60)$$

We have the following

Proposition 5.3. *Given $z \in Z^1(K, G)$, the following assertions hold:*

(a) *given $g \in \mathcal{G}(z)$, then $\alpha_g(u) \in U^1(K, z)$ for any $u \in U^1(K, z)$;*

(b) *The mapping*

$$\alpha : \mathcal{G}(z) \times U^1(K, z) \ni (g, u) \longrightarrow \alpha_g(u) \in U^1(K, z) \quad (61)$$

defines a left action, not free, of $\mathcal{G}(z)$ on $U^1(K, z)$.

Proof. (a) Clearly $\alpha_g(u)(\bar{b}) = \alpha_g(u)(b)^{-1}$ for any 1-simplex b . Moreover, if $b \in \Sigma_1^{\text{inf}}(K)$, then $\alpha_g(u)(b) = g_{\partial_0 b} u(b) g_{\partial_1 b}^{-1} = g_{\partial_0 b} z(b) g_{\partial_1 b}^{-1} = z(b)$. This entails that $\alpha_g(u)$ satisfies property (ii) of the definition of connections. Hence $\alpha_g(u) \in U^1(K, z)$. (b) Clearly, α is a left action that is not free, because $z \in U^1(K, z)$, hence $\alpha_g(z) = z$ for any $g \in \mathcal{G}(z)$. \square

6 Conclusions and outlook

We have developed a theory of bundles over posets from a cohomological standpoint, the analogue of describing the usual principal bundles in terms of their transition functions. In a sequel, we will introduce principal bundles over posets and their mappings directly and further develop such concepts as connection, curvature, holonomy and transition function (we will also introduce concepts such as gauge group and gauge transformation). Although all these concepts are familiar from the usual theory of principal bundles, at this point it is worth stressing some of the differences from that theory. As we shall see in the sequel, the definition of principal bundle involves bijections between different fibres satisfying a 1-cocycle identity. An important rôle is played by the simplicial set of inflationary simplices. All principal bundles can be trivialized on the fundamental covering. Finally, it should be stressed that the goal of these investigations is to develop gauge theories in the framework of algebraic quantum field theory. Our principal fibre bundles and the associated vector bundles are envisaged stepping stones to the algebra of observables.

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